Plasma Measurement I

Ion Probes

Ion Flux Equation

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Outline

- Integration of constant
- Ion flux equation
- Spacecraft potential
- Arrival angles

Integration of constant

Assuming a collision-less, electrostatic and no external magnetic field condition, the Vlasov equation can be rewritten as

$$\frac{\partial f_j}{\partial t} + \mathbf{v} \cdot \frac{\partial f_j}{\partial \mathbf{x}} + \frac{q_j}{m_j}\left(\mathbf{E} + \frac{\mathbf{v} \times \mathbf{B}}{c}\right) \cdot \frac{\partial f_j}{\partial \mathbf{v}} = 0$$

The phase space distribution function will be conserved if follows a dynamical trajectory described as

$$\frac{dx}{\upsilon_x} = \frac{dy}{\upsilon_y} = \frac{dz}{\upsilon_z} = \frac{d\upsilon_x}{-\frac{q_j}{m_j}\frac{\partial V}{\partial x}} = \frac{d\upsilon_y}{-\frac{q_j}{m_j}\frac{\partial V}{\partial y}} = \frac{d\upsilon_z}{-\frac{q_j}{m_j}\frac{\partial V}{\partial z}}$$

An integration of constant (conservation law of total energy) can be obtained as

$$\frac{m_j \upsilon^2}{2} + q_j V = const$$

Maxwell-Boltzmann distribution function

Any function with the energy conservation form will be accepted as a solution of the Vlasov equation, so the distribution function can be described as

$$f(\mathbf{x},\mathbf{v}) = f\left(\frac{m\upsilon^2}{2} + qV(\mathbf{x})\right)$$

In our application, a satellite flows through the ionosphere horizontally with a supersonic speed, a three-dimensional Maxwell-Boltzmann distribution function with drift velocity *U* is applied to obtain the steady flux collected within the retarding potential analyzer as

$$f(\mathbf{x},\mathbf{v}) = n_o\left(\frac{m}{2\pi\kappa T}\right)^{\frac{3}{2}} \exp\left[-\left(\sqrt{\frac{m\upsilon^2}{2\kappa T} + \frac{qV(\mathbf{x})}{\kappa T}} - \sqrt{\frac{mU^2}{2\kappa T}}\right)^2\right]$$

$$f(\mathbf{x},\mathbf{v}) = n_o\left(\frac{m}{2\pi\kappa T}\right)^{\frac{3}{2}} \exp\left[-\left(\frac{m(\upsilon-U)^2}{2\kappa T} + \frac{qV(\mathbf{x})}{\kappa T}\right)\right]$$

The moment equations

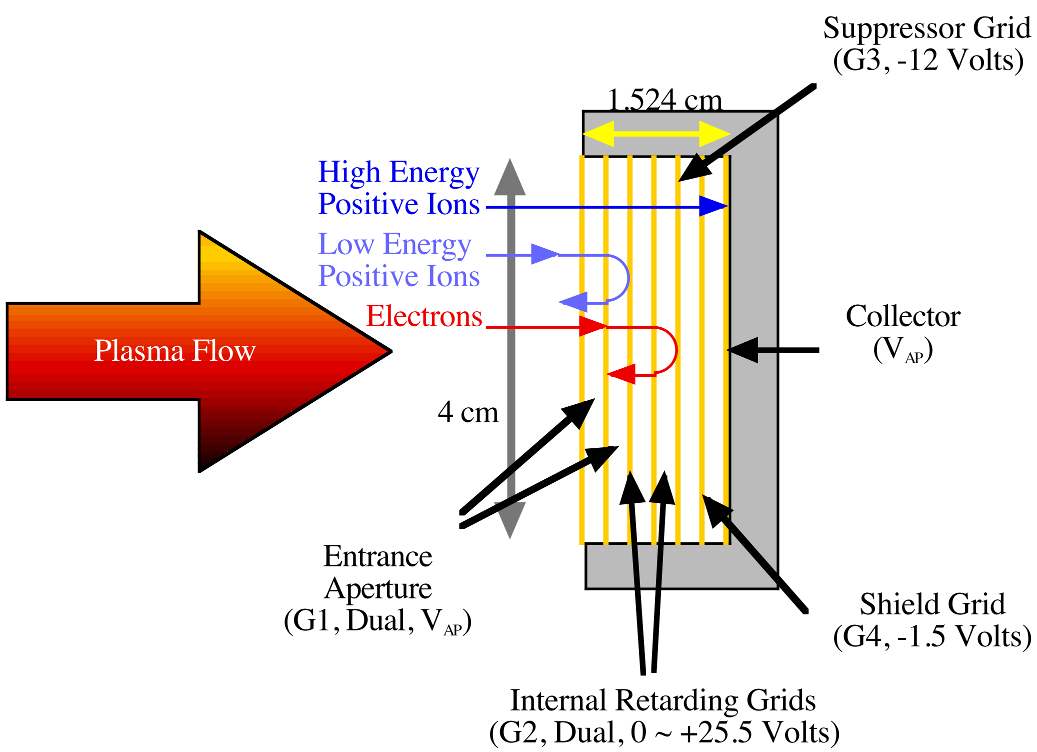
The particle number density *n* and flux Φ are defined by integrating the phase space distribution function over velocity space as

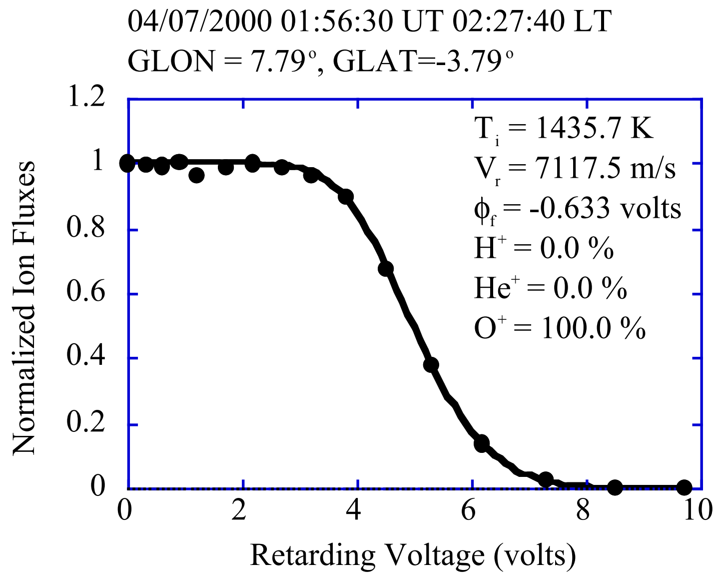
$$n(\mathbf{x}) \equiv \int f(\mathbf{x},\mathbf{v})\, d\mathbf{v}$$

$$\Phi(\mathbf{x}) \equiv n(\mathbf{x})\mathbf{u}(\mathbf{x}) \equiv \int f(\mathbf{x},\mathbf{v})\,\mathbf{v}\, d\mathbf{v}$$

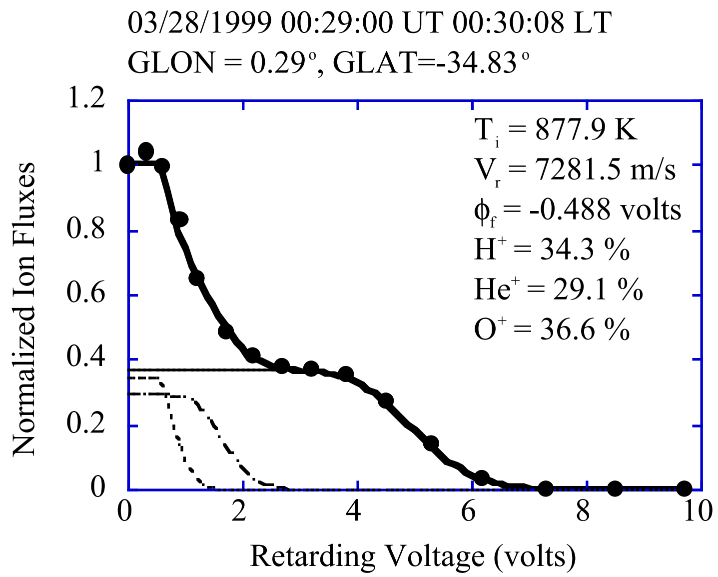
where **u**(**x**) is the average velocity of incoming plasma at position **x**. It is noted that lower and upper velocity limits become important in deriving these macroscopic density and flux for different boundary conditions.

Retarding potential analyzer (RPA)





O^+ only

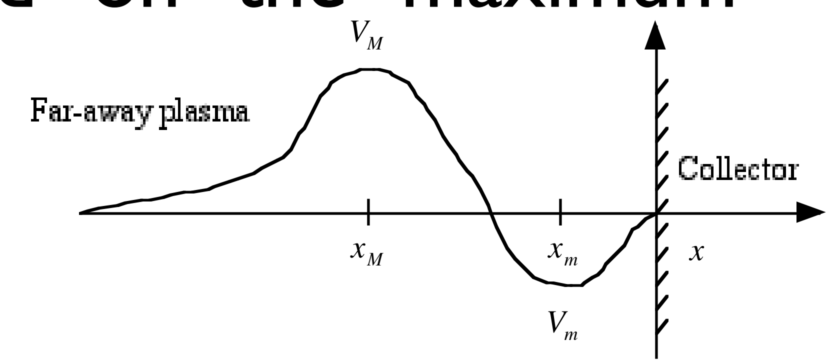


Light ions and O^+

Ion temperature, composition, ram speed, and floating potential

The lower velocity limit

To derive the ion flux within the RPA, several assumptions are made here. We set electric potential to zero for plasma far away from collector. In a fixed frame of a satellite rigid body, positive ions can stream with about satellite velocity from outside the RPA into the collector. Based on the conservation of energy, the retarding potential structure inside the RPA will prevent the low energy positive ions from reaching the collector. The lower velocity limit v_L for integration is depended on the maximum potential V_M along the streaming path and its location



$$\upsilon_L = \pm\sqrt{\frac{2q\left[V_M - V(x)\right]}{M}}$$

where M is the ion mass. The positive sign indicates that the distribution has overcome the maximum potential barrier (between the retarding grid and the collector, i.e., $x > x_M$) and the negative sign indicates that the distribution has return particles (between the far away space and retarding grid, i.e., $x < x_M$).

The ion flux equation

The flux for a single species can be integrated as

$$\Phi(x) = \int_{\upsilon_L}^{\infty} f\upsilon\, d\upsilon = \int_{\pm\sqrt{\frac{2q[V_M - V(x)]}{M}}}^{\infty} n_o \sqrt{\frac{M}{2\pi\kappa T}} \exp\left[-\left(\sqrt{\frac{M\upsilon^2}{2\kappa T} + \frac{qV}{\kappa T}} - \sqrt{\frac{MU^2}{2\kappa T}}\right)^2\right] \upsilon\, d\upsilon$$

Let's define the variable α and substitute it into the flux equation

$$\alpha = \sqrt{\frac{M\upsilon^2}{2\kappa T} + \frac{qV}{\kappa T}} - \sqrt{\frac{MU^2}{2\kappa T}} \rightarrow \upsilon d\upsilon = \frac{\alpha + \sqrt{\frac{MU^2}{2\kappa T}}}{\left(\frac{M}{2\kappa T}\right)} d\alpha$$

The lower limit of the integration is not a function of position any more and can be changed to

$$\alpha_L = \sqrt{\frac{qV_M}{\kappa T}} - \sqrt{\frac{MU^2}{2\kappa T}}$$

The ion flux equation (cont.)

The flux can be modified as

$$\Phi = \int_{\sqrt{\frac{qV_M}{\kappa T}}-\sqrt{\frac{MU^2}{2\kappa T}}}^{\infty} n_o \sqrt{\frac{M}{2\pi\kappa T}} \exp\left[-\alpha^2\right]\left(\alpha + \sqrt{\frac{MU^2}{2\kappa T}}\right)\frac{d\alpha}{\left(\frac{M}{2\kappa T}\right)}$$

Using these definite integrals from mathematical handbook

$$\int_x^{\infty} \exp\left[-\alpha^2\right] d\alpha = \frac{\sqrt{\pi}}{2}\left[1 - erf\left(x\right)\right] \qquad \int_x^{\infty} \exp\left[-\alpha^2\right]\alpha \, d\alpha = \frac{\exp\left[-x^2\right]}{2}$$

The flux equation can be derived as

$$\Phi\left(V_M\right) = n_o U \left\{ \frac{1}{2}\left[1 + erf\left(\beta F\right) + \frac{\exp\left[-\left(\beta F\right)^2\right]}{\sqrt{\pi}\beta U} \right] \right\}$$

, where

$$\beta = \sqrt{\frac{M}{2\kappa T}} \qquad F = U - \sqrt{\frac{2qV_M}{M}}$$

$$\beta F = \frac{U}{\sqrt{\frac{2\kappa T}{M}}} - \sqrt{\frac{qV_M}{\kappa T}}$$

The ion flux equation (cont.)

The derived ion flux [*Whipple*, 1959] is determined by the maximum electric potential and is a constant value from the far away plasma to the collector of the RPA once a maximum electric potential is known. It makes sense for a steady state condition that we setup for the ion flux model.

For multiple species, the ion flux equation can be generalized as

$$\Phi(V_M) = \sum_i \Phi_i(V_M) = \sum_i n_{oi} U_i \left\{ \frac{1}{2} \left[1 + erf(\beta_i F_i) + \frac{\exp\left[-(\beta_i F_i)^2\right]}{\sqrt{\pi}\beta_i U_i} \right] \right\}$$

where $\beta_i = \sqrt{\frac{M_i}{2\kappa T_i}}$, $F_i = U_i - \sqrt{\frac{2q_i V_M}{M_i}}$, and $\beta_i F_i = \frac{U_i}{\sqrt{\frac{2\kappa T_i}{M_i}}} - \sqrt{\frac{q_i V_M}{\kappa T_i}}$

High ram velocity

From the ion flux equation for multiple species, the ion flux at far away plasma (the electric potential is set to zero) is expressed as

$$\Phi(0)=\sum_i \Phi_i(0)=\sum_i n_{oi}U_i\left\{\frac{1}{2}\left[1+erf\left(\beta_i U_i\right)+\frac{\exp\left[-\left(\beta_i U_i\right)^2\right]}{\sqrt{\pi}\beta_i U_i}\right]\right\}$$

For ROCSAT observation, the ion ram velocities U_i (~ satellite velocity) are larger than the ion thermal speeds, i.e.:

$$\beta_i U_i = U_i / \sqrt{\frac{2\kappa T_i}{M_i}} >> 1$$

The error function can be expanded for large $\beta_i U_i$ and the ion flux at zero electric potential can be reduced to

$$\Phi(0)=\sum_i \Phi_i(0)\sim\sum_i n_{oi}U_i\left\{\frac{1}{2}\left[1+1-\frac{\exp\left[-\left(\beta_i U_i\right)^2\right]}{\sqrt{\pi}\beta_i U_i}\left(1-\frac{1}{2\beta_i^2 U_i^2}+...\right)+\frac{\exp\left[-\left(\beta_i U_i\right)^2\right]}{\sqrt{\pi}\beta_i U_i}\right]\right\}\sim\sum_i n_{oi}U_i$$

High ram velocity (cont.)

where $\Phi(0)$ and $\Phi_i(0)$ are the known parameters and can be estimated from the I-V data. It is noted that this approximation can be applied for most of satellite observations, but not for sounding rocket observations. That is because the sounding rocket has an orbital velocity with similar order of magnitude as the ion thermal velocity.

Cut-off potential

We can define a cut-off electric potential V_{ci} for species *i* to make $\beta_i F_i = 0$. It is similar to zero point of the second derivative of the I-V curve with respect to the potential mentioned by *Greenspan* et al. [1986]. Such the cut-off potential can be written as

$$V_{ci} \equiv \frac{M_i U_i^2}{2q_i}$$

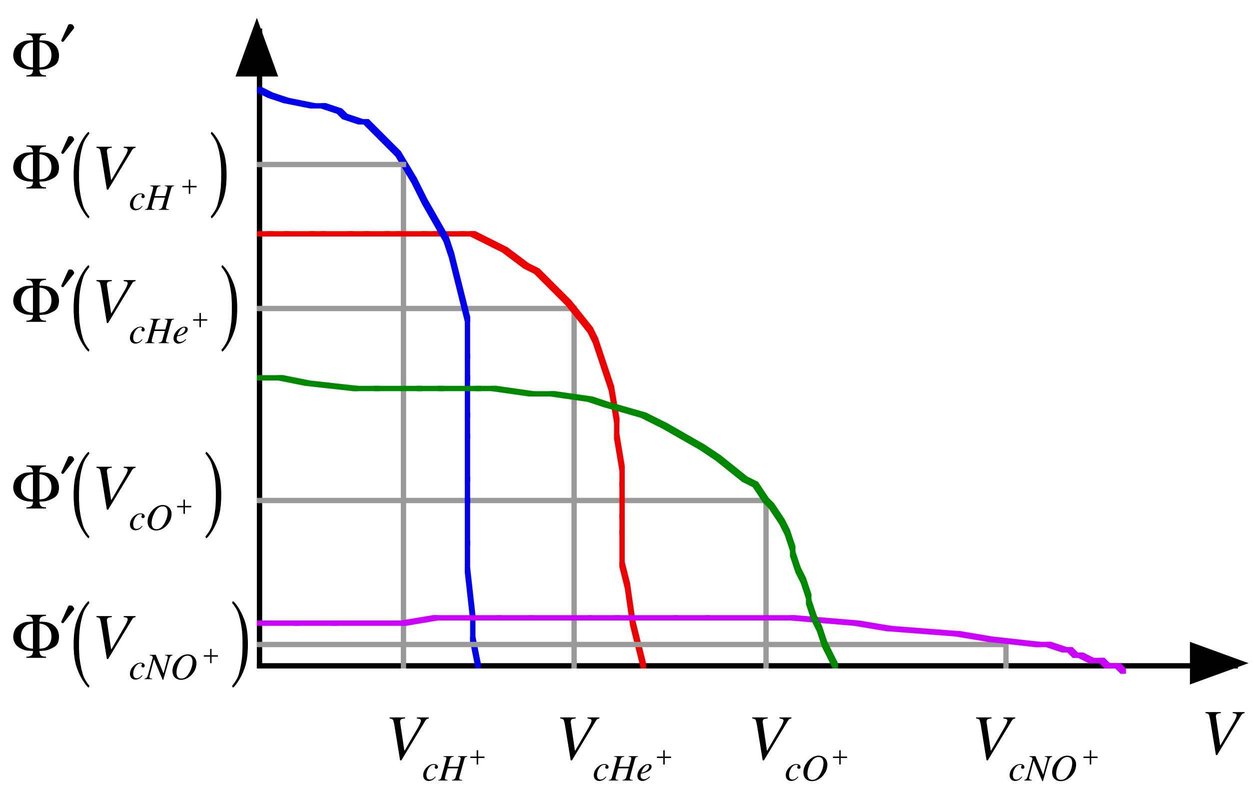
The **cut-off potential** was defined [*Chao and Su*, 1999] as the potential that prevents the mean flow of a specific species from reaching the collector plate. The ion fluxes can be written as

$$\Phi_j\left(V_{ci}\right) \sim \begin{cases} 0, & \text{if } i > j \\ \Phi_i(0)\left\{\frac{1}{2}\left[1+\frac{1}{\sqrt{\pi}\sqrt{\frac{q_i V_{ci}}{\kappa T_i}}}\right]\right\} = \Phi_i(0)\left\{\frac{1}{2}\left[1+\frac{1}{\sqrt{\pi}\sqrt{\frac{M_i U_i^2}{2\kappa T_i}}}\right]\right\}, & \text{if } i = j \\ \Phi_j(0), & \text{if } i < j \end{cases}$$

Cut-off potential (cont.)

where the subscript *i* and *j* are both denoted for species and are ordered with atomic mass from small to large. Therefore the total ion flux can be written as

$$\Phi(V_{ci}) = \sum_j \Phi_j(V_{ci}) \sim \Phi_i(0)\left\{\frac{1}{2}\left[1+\frac{1}{\sqrt{\pi}\sqrt{\frac{q_i V_{ci}}{\kappa T_i}}}\right]\right\} + \sum_{i<j}\Phi_j(0) = \Phi_i(0)\left\{\frac{1}{2}\left[1+\frac{1}{\sqrt{\pi}\sqrt{\frac{M_i U_i^2}{2\kappa T_i}}}\right]\right\} + \sum_{i<j}\Phi_j(0)$$



Half current approximation

Considering the ROCSAT flying speed and the ionospheric environment at 600 km altitude, most of cases that we study indicate $M_i U_i^2/2 > \kappa T_i$. Therefore the total ion flux $\Phi(V_{ci})$ can be approximated as

Half current at cut-off potential

$$\Phi(V_{ci}) = \sum_j \Phi_j(V_{ci}) \sim \boxed{\frac{\Phi_i(0)}{2}} + \sum_{i<j} \Phi_j(0)$$

Once the $\Phi(V_{ci})$ are estimated, the cut-off potentials V_{ci} can be searched from the I-V data measured by the RPA through interpolation methods and are expressed as

$$V_{ci} \sim V\left(\Phi = \frac{1}{2}\Phi_i(0) + \sum_{i<j} \Phi_j(0)\right)$$

The ram velocity and the ion density for each species can be derived from

$$U_i = \sqrt{\frac{2 q_i V_{ci}}{M_i}}$$

$$n_{oi} = \frac{\Phi_i(0)}{U_i \left\{ \frac{1}{2}\left[1 + erf(\beta_i U_i) + \frac{\exp\left[-(\beta_i U_i)^2\right]}{\sqrt{\pi}\beta_i U_i} \right] \right\}} \sim \frac{\Phi_i(0)}{U_i}$$

Half current approximation (cont.)

To derive the ion temperature, we can differentiate the ion flux equation for species *i* with electric potential

$$\frac{d\Phi_i}{dV} = \frac{q_i n_{oi}}{\sqrt{2\pi}} \frac{\exp\left[-(\beta_i F_i)^2\right]}{\sqrt{M_i \kappa T_i}}$$

The slope of *j*-ion flux at V_{ci} can be approximated as

$$\frac{d\Phi_j}{dV}(V_{ci}) \sim \begin{cases} 0, & \text{if } i \neq j \\ -\dfrac{q_i \Phi_i(0)}{U_i \sqrt{2\pi M_i \kappa T_i}}, & \text{if } i = j \end{cases}$$

Therefore the slope of the total ion flux at V_{ci} can be estimated as

$$\frac{d\Phi}{dV}(V_{ci}) = \sum_j \frac{d\Phi_j}{dV}(V_{ci}) = -\frac{q_i \Phi_i(0)}{U_i \sqrt{2\pi M_i \kappa T_i}}$$

Half current approximation (cont.)

Once the dΦ*(V_{ci})/dV* is known (also available from the interpolation of the I-V data), the ion temperature for species *i* can then be calculated as

$$\kappa T_i = \frac{q_i^2 \Phi_i^2(0)}{2\pi M_i U_i^2 \left[\frac{d\Phi}{dV}(V_{ci})\right]^2} = \frac{q_i \Phi_i^2(0)}{4\pi V_{ci} \left[\frac{d\Phi}{dV}(V_{ci})\right]^2}$$

Hence the half-current approximation can obtain the ion ram velocity, ion density and ion temperature for each species. These results can be served as initial guesses for a more sophisticate regression fit.

Zero ram velocity

The ion flux equation can be degenerated as

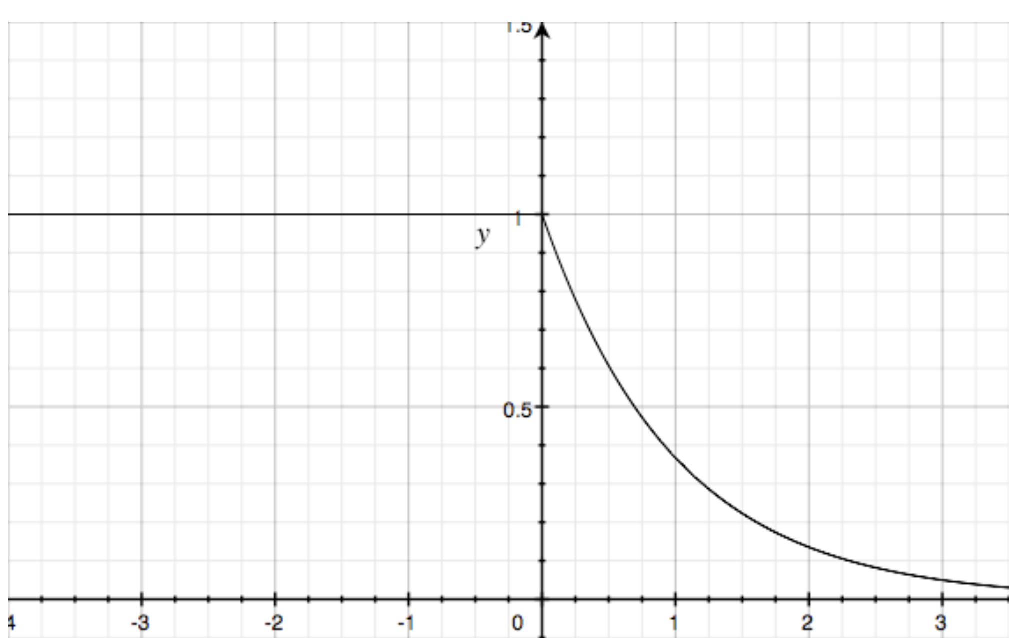
$$\Phi(V_M) = \sum_i \Phi_i(V_M) = \sum_i n_{oi}\sqrt{\frac{\kappa T_i}{2\pi M_i}}\exp\left[-\frac{q_i V_M}{\kappa T_i}\right]$$

The ion temperature can be obtained from the slope of I-V curve.

$$I = A\Phi = An_o\sqrt{\frac{\kappa T_i}{2\pi M_i}}\exp\left(-\frac{eV_M}{\kappa T_i}\right)$$

$$\ln(I) = \ln\left[An_o\exp\left(-\frac{eV_M}{\kappa T_i}\right)\sqrt{\frac{\kappa T_i}{2\pi M_i}}\right] = \ln\left[An_o\sqrt{\frac{\kappa T_i}{2\pi M_i}}\right] - \frac{eV_M}{\kappa T_i}$$

$$\frac{d}{dV_M}\left[\ln(I)\right] \sim -\frac{e}{\kappa T_i}$$



Spacecraft potential

Assuming the plasma is streaming into the collector of the RPA with a velocity about spacecraft flying speed, the ion flux for species *i* can be approximated as

$$\Phi_i = \frac{1}{4} n_{oi} U_i$$

For electron flux at the spacecraft skin is given by

$$\Phi_e = \frac{1}{4} n_e \overline{\upsilon}_e = \frac{1}{4} n_o \exp\left[\frac{eV}{\kappa T_e}\right] \sqrt{\frac{8\kappa T_e}{\pi m}}$$

Because the electron thermal speed is extremely larger than the satellite flying speed, the electron flux is not related to the satellite flying speed. Once the zero net current flow condition is satisfied

$$I = 0 = A\left(\sum_i q_i \Phi_i - e\Phi_e\right) = A\left\{\sum_i \frac{1}{4} q_i n_{oi} U_i - e\frac{1}{4} n_o \exp\left[\frac{eV_f}{\kappa T_e}\right]\sqrt{\frac{8\kappa T_e}{\pi m}}\right\}$$

Spacecraft potential (cont.)

For plasma with electron and a single-charged ion species, the floating potential of moving object model can be written as

$$V_f = -\frac{\kappa T_e}{2e}\ln\left(\frac{8\kappa T_e}{\pi m U_i^2}\right)$$

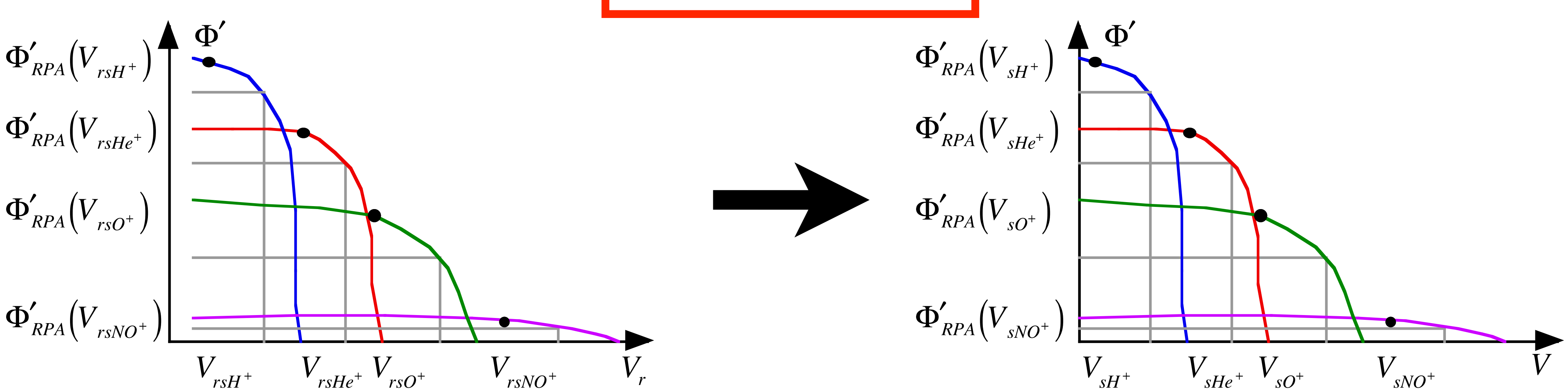
It is the same floating potential written by *Zuccaro and Holt* [1982] for a sensor onboard a moving spacecraft environment. For plasma with electron and multiple-charged ion species, the floating potential of moving object model can be written as

$$V_f = -\frac{\kappa T_e}{2e}\ln\left(\frac{8\kappa T_e}{\pi m U_s^2 \sum_i q_i' n_i' U_i'^2}\right), \text{ where } q_i' = \frac{q_i}{e}, n_i' = \frac{n_{oi}}{n_o}, \text{ and } U_i' = \frac{U_i}{U_s}$$

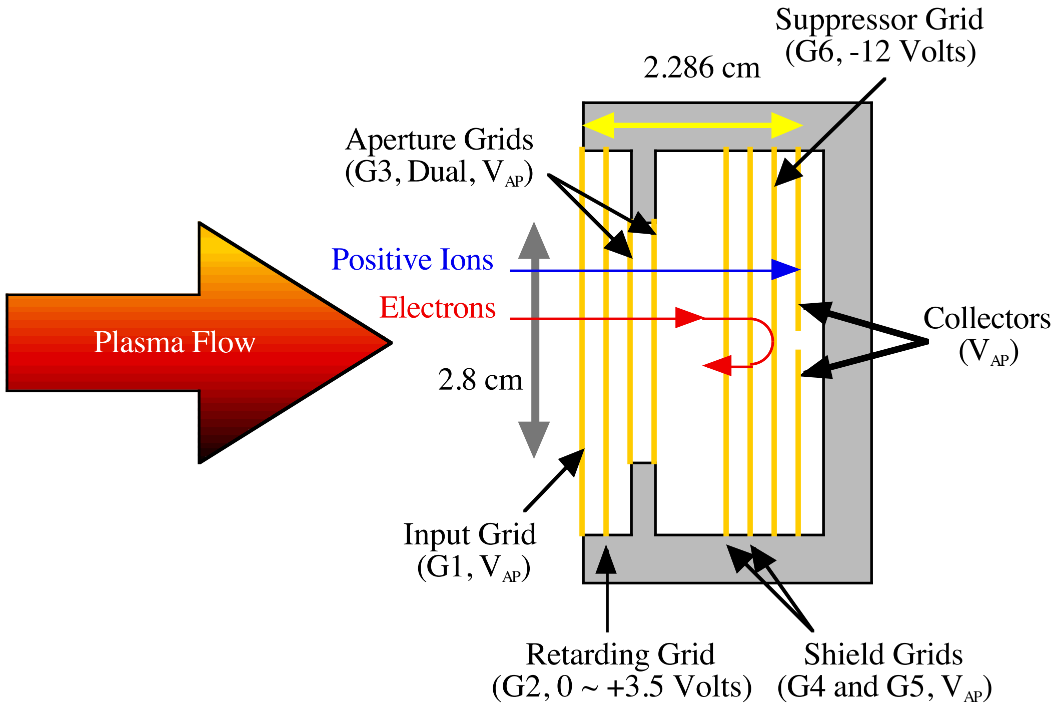
Retarding potential

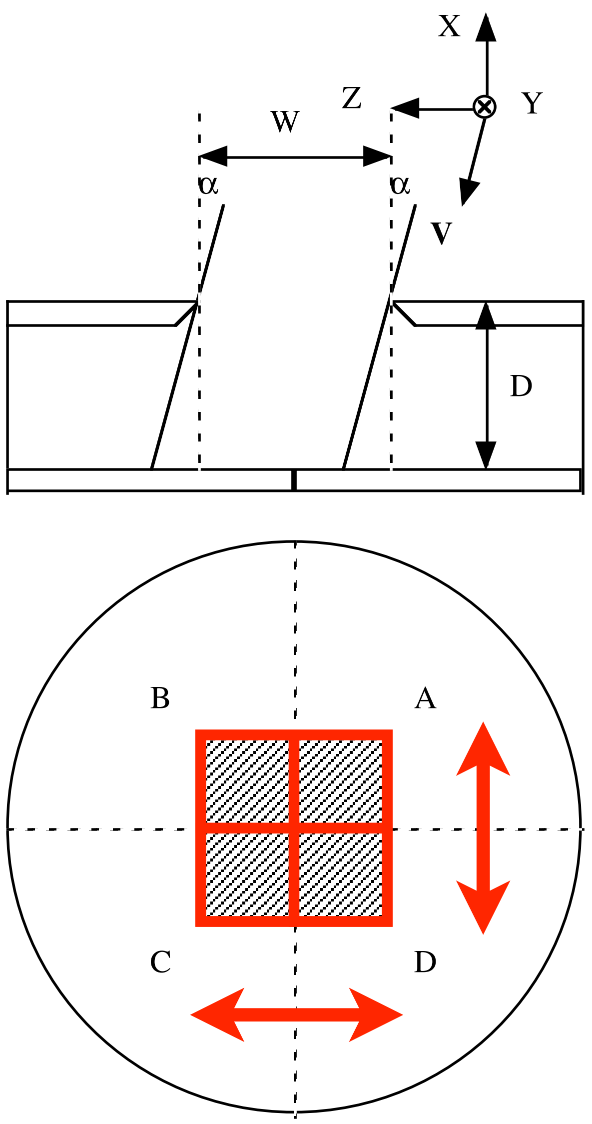
In the formulation, all the potentials are reference to plasma potential. The plasma potential is assumed to be zero. However, all the sensors of IPEI instrument are referenced to a senpot circuit and the circuit is to maintain a reference potential to approach the plasma potential (in actual, the reference potential is to approach the floating potential) as possible [*Zuccaro and Holt*, 1982]. Because the retarding voltage (V_r) is referenced to the floating potential (V_f), the retarding potential (V) (referenced to far-away plasma) can be express as

$$V = V_r + V_f$$



Ion drift meter (IDM)



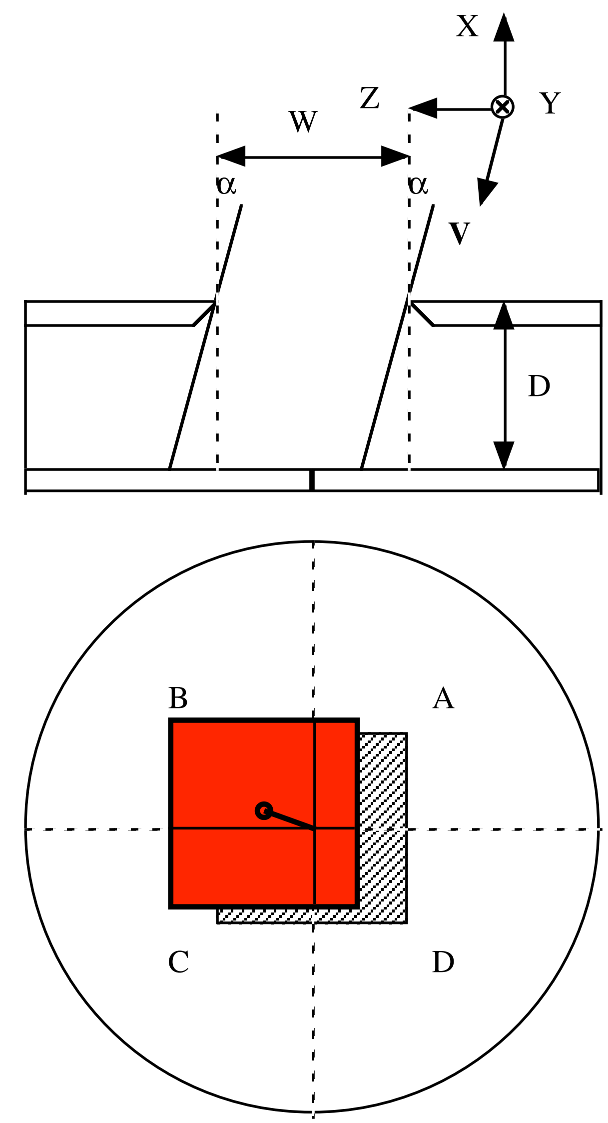


Arrival angles of ion flow

Ion velocity in ICS

Angles of arrival from current difference of two collectors

$$\frac{\frac{W}{2}+D\tan\alpha}{\frac{W}{2}-D\tan\alpha}=\frac{I_{BC}}{I_{AD}}=\frac{I_1}{I_2}\rightarrow\tan\alpha=\left(\frac{W}{2D}\right)\left(\frac{I_1-I_2}{I_1+I_2}\right)$$



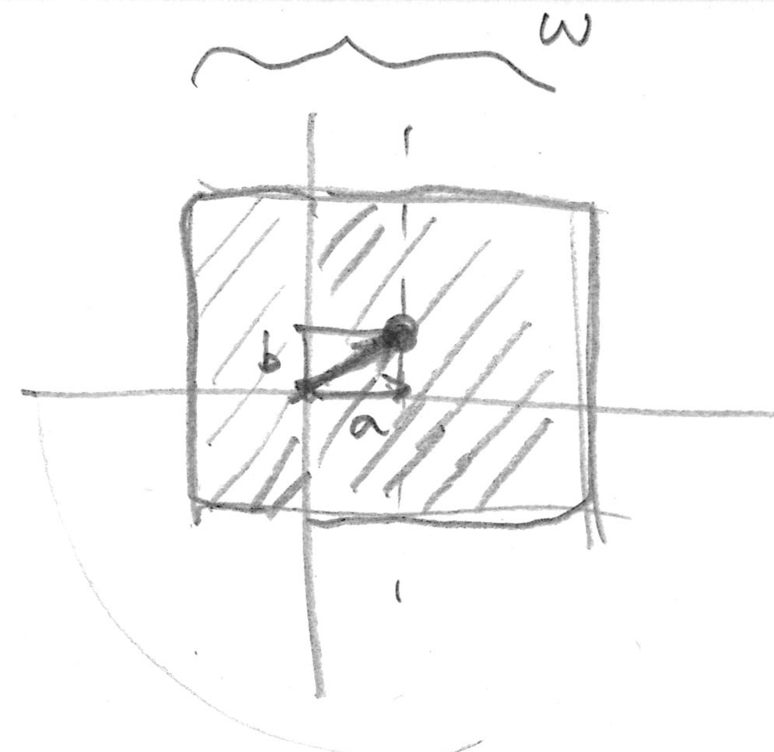
The velocity components at Y and Z directions are:

$$\upsilon_y=U_i\tan\alpha_y=(-\upsilon_x)\tan\alpha_y$$

$$\upsilon_z=U_i\tan\alpha_z=(-\upsilon_x)\tan\alpha_z$$

$$\frac{a}{D} = \tan\alpha$$

$$\frac{b}{D} = \tan\beta$$



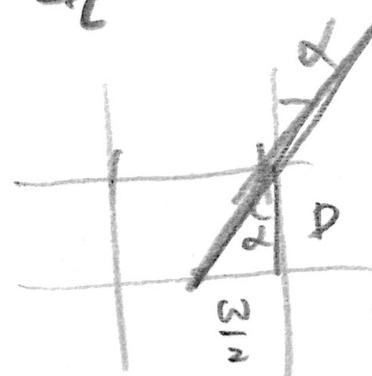
$$\frac{\frac{\omega}{2}+a}{\frac{\omega}{2}-a} = \frac{I_1}{I_2}$$

$$\frac{\omega \times \left(\frac{\omega}{2}+D\tan\alpha\right)}{\omega \times \left(\frac{\omega}{2}-D\tan\alpha\right)} = \frac{I_1}{I_2} \rightarrow$$

$$\frac{\omega}{2}+D\tan\alpha = \left(\frac{I_1}{I_2}\right)\left(\frac{\omega}{2}-D\tan\alpha\right)$$

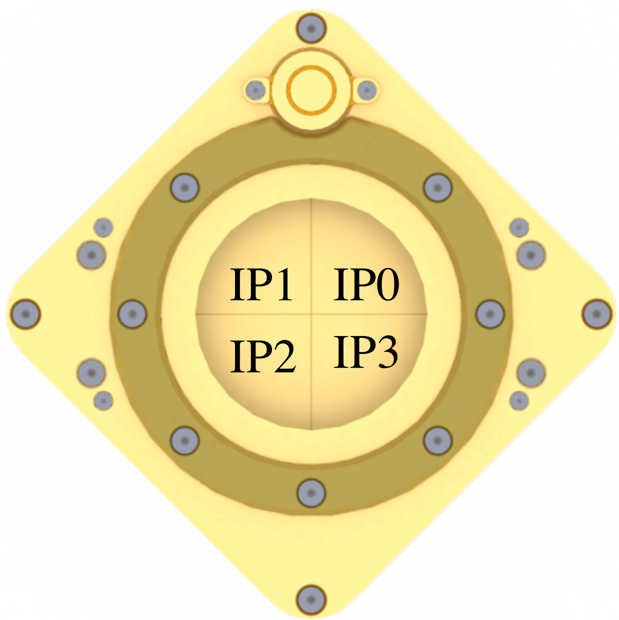
$$\left(1+\frac{I_1}{I_2}\right) D\tan\alpha = \left(\frac{I_1}{I_2}-1\right)\frac{\omega}{2}$$

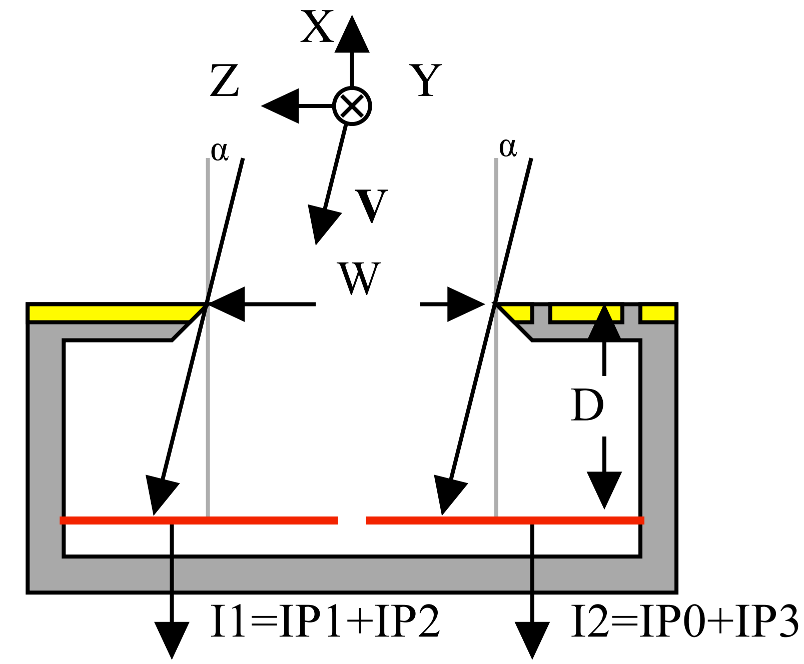
$$\tan\alpha = \frac{\omega}{2D}\,\frac{\left(\frac{I_1}{I_2}-1\right)}{\left(1+\frac{I_1}{I_2}\right)} = \frac{\omega}{2D}\left(\frac{I_1-I_2}{I_1+I_2}\right)$$



$$-1 < \frac{I_1-I_2}{I_1+I_2} < 1$$

IDM



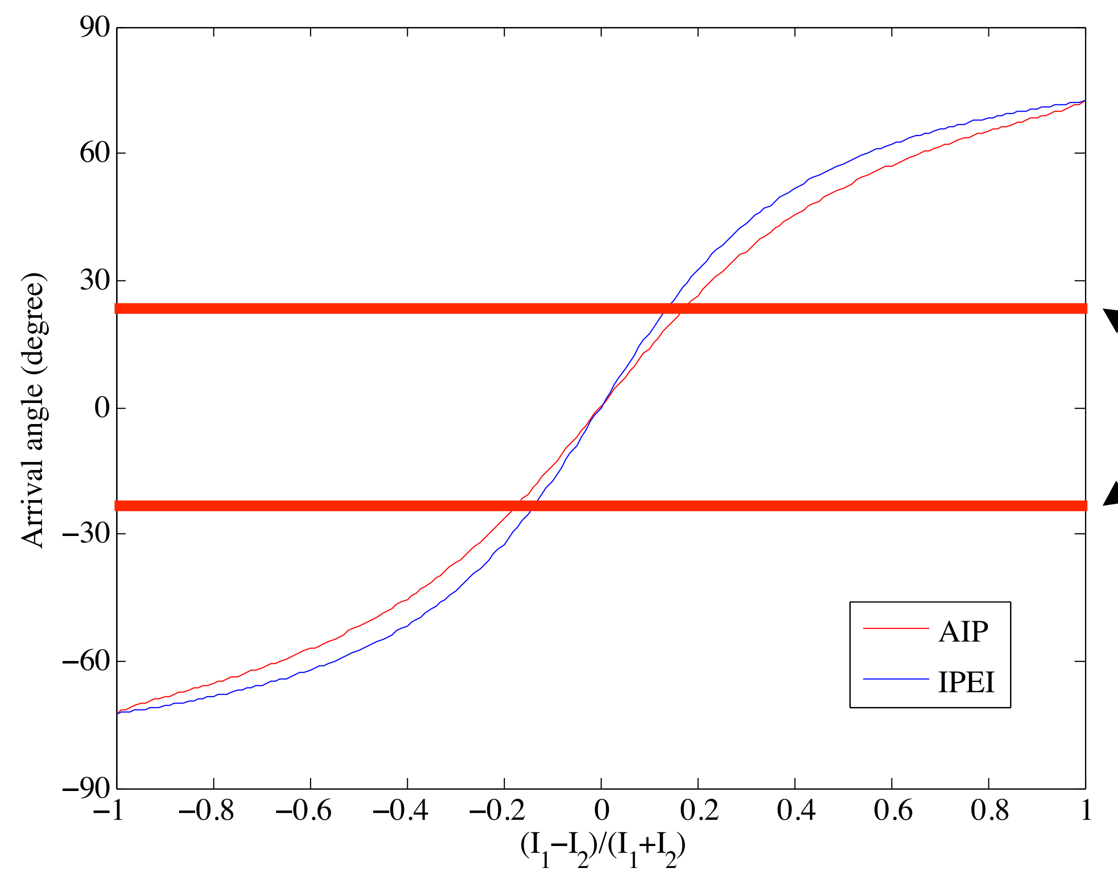


For a circular opening in AIP

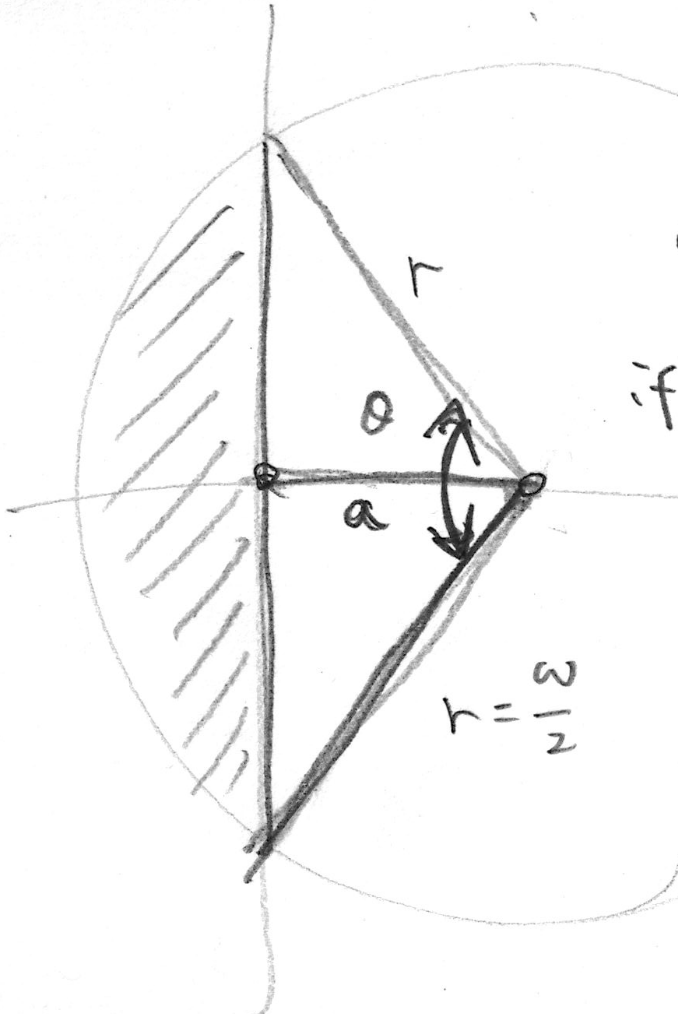
$$\tan\alpha = \frac{W}{2D}\cos\frac{\theta}{2}, \quad \theta = \sin\theta + \left[1 - \left(\frac{I_1 - I_2}{I_1 + I_2}\right)\right]\pi$$

For a square opening in IPEI

$$\tan\alpha = \frac{W}{2D}\left(\frac{I_1 - I_2}{I_1 + I_2}\right)$$



FOV

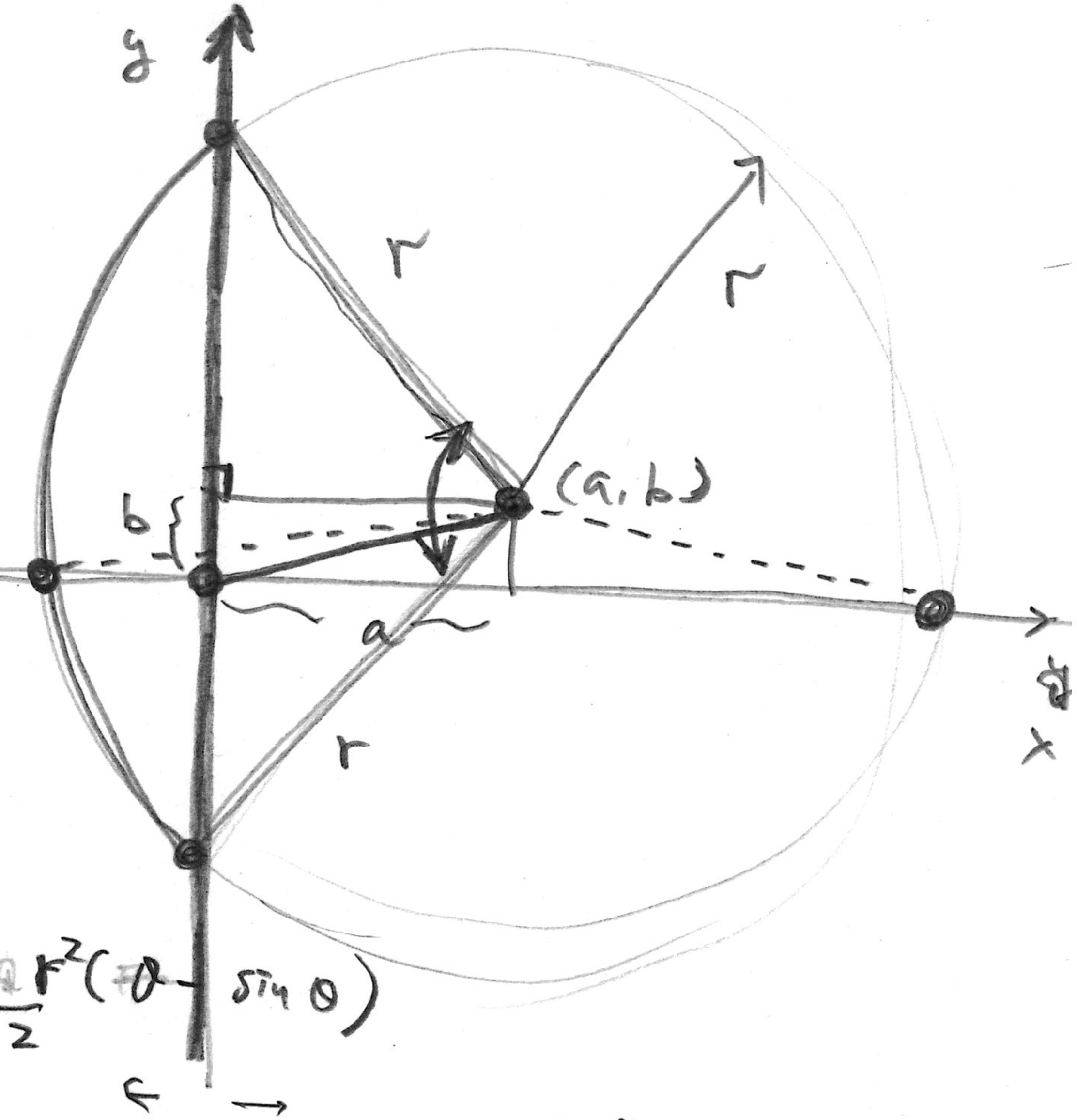


$$\cos\left(\frac{\theta}{2}\right) = \frac{a}{r}$$

if $a \to 0$ $\frac{\theta}{2} \to \frac{\pi}{2}$

$\theta \to \pi$

$\theta : \to \frac{\theta}{2} r^2$



$L = 2 \cdot r \cdot \sin\frac{\theta}{2}$

$a = r \cdot \cos\frac{\theta}{2}$

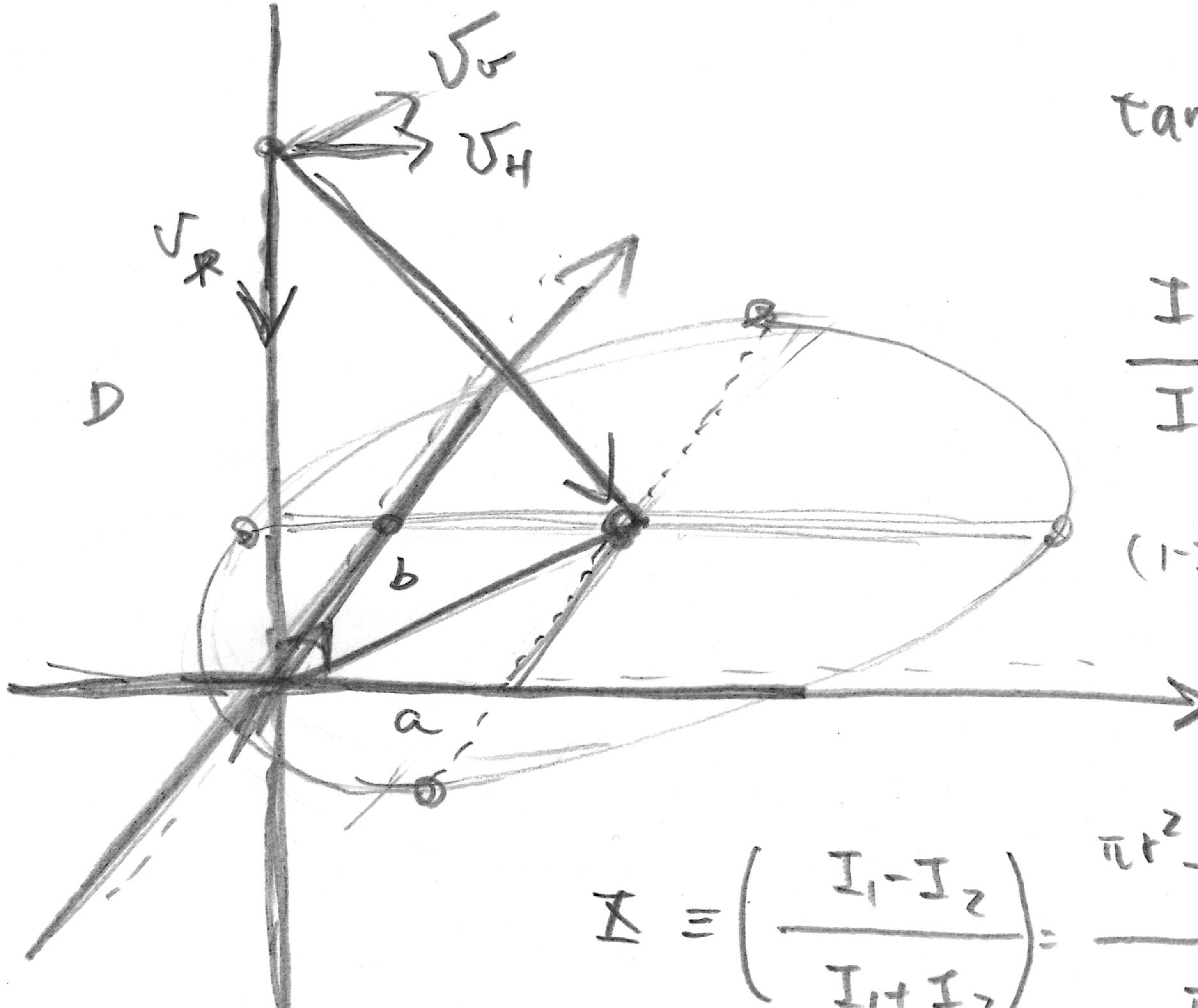
$$\frac{\theta}{2} r^2 - \frac{1}{2} a \cdot L$$

$$= \frac{\theta}{2} r^2 - \frac{1}{2} \cdot r\cos\frac{\theta}{2} \cdot 2 \cdot r \sin\frac{\theta}{2} = \frac{1}{2} r^2 (\theta - \sin\theta)$$

$a \to 0$

$$I_2 = \frac{1}{2} r^2 (\theta - \sin\theta) \;;\; \pi r^2 - \frac{1}{2} r^2 (\theta - \sin\theta) = I_1$$

$$\therefore \frac{I_1}{I_2} = \frac{\pi r^2 - \frac{1}{2} r^2 (\theta - \sin\theta)}{\frac{1}{2} r^2 (\theta - \sin\theta)} = \frac{2\pi - (\theta - \sin\theta)}{\theta - \sin\theta}$$



$$\tan\alpha = \frac{a}{D}$$

$$\tan\beta = \frac{b}{D}$$

$$\frac{I_1}{I_2} = \frac{2\pi - (\theta - \sin\theta)}{\theta - \sin\theta}$$

$(1-X)\pi$

$$X \equiv \left(\frac{I_1 - I_2}{I_1 + I_2}\right) = \frac{\pi r^2 - \frac{1}{2} r^2(\theta - \sin\theta) - \frac{1}{2} r^2(\theta - \sin\theta)}{\pi r^2}$$

$$\cos\frac{\theta}{2} = \frac{a}{\frac{\omega}{2}} = \frac{D\tan\alpha}{\frac{\omega}{2}}$$

$$\pi X = (\pi - \theta + \sin\theta)$$

$$(1-X)\pi - \theta + \sin\theta = 0$$

$$= 1 - \left(\frac{\theta - \sin\theta}{\pi}\right)$$

$$\tan\alpha = \frac{\omega}{2D}\cos\frac{\theta}{2}$$

$$\frac{\theta}{2} = \frac{(1-X)\pi + \sin\theta}{2}$$

(θ starts from π to 0 as a from 0 to $\frac{\omega}{2}$)